# WILSON'S EXPANSION WITH POWER ACCURACY*


Xiangdong Ji

*Center for Theoretical Physics*
*Laboratory for Nuclear Science*
*and Department of Physics*
*Massachusetts Institute of Technology*
*Cambridge, Massachusetts 02139*



(MIT-CTP-2437 hep-ph/xxxxxxx June 1995)



## Abstract

Because of the infrared renormalons, it is difficult to get power accuracy in the traditional approach to the Wilson's operator product expansion. Based on a new perturbative renormalization scheme for power-divergent operators, I propose a practical version of the OPE that allows to calculate power corrections to desired accuracy. The method is applied to the expansion of the vector-current correlation function in QCD vacuum, in which field theoretical status of the gluon condensate is discussed.





Typeset using REVTEX



*This work is supported in part by funds provided by the U.S. Department of Energy (D.O.E.) under cooperative agreement #DF-FC02-94ER40818.

Wilson's operator product expansion (OPE) [1] has become one of the most powerful tools in modern field theory. Indeed, many graduate text books on field theory devote an entire chapter treating the subject [2]. Direct applications of the OPE in Quantum Chromodynamics (QCD) include deep-inelastic scattering, heavy-quark expansion, weak-nonleptonic decays, and QCD sum rule calculations, etc. Generally speaking, the OPE provides the starting-point for any perturbative calculations in hard processes. The underlying principle of the OPE—scale separation and factorization— has spawned many uses of the so-called effective field theories in nuclear and particle physics.

Given the success of the OPE, it may be surprising to realize that the standard text-book approach to the expansion is too formal to be used in some applications. A problem occurs when one seriously considers the so-called *power corrections* to the leading term in the expansion. To explain, let me first remind the reader some basics of Wilson's expansion [2]. Consider a time-ordered product of two currents separated by, say, an Euclidean short distance $\xi^2$. According to Wilson, the product can be expanded as,

$$T[J(\xi)J(0)] = \sum_{i=0}^{\infty} C_i(\xi^2, \mu^2) O_i(\mu^2) \ , \tag{1}$$

where, for simplicity, I have neglected possible Lorentz indices. The short distance physics above the factorization scale $\mu^2$ is included in the coefficient functions $C_i$, ordered according to the descending singularities as $\xi^2 \to 0$. $O_i(\mu^2)$ are renormalized operators with increasing dimensions, designed to account for long distance physics below $\mu^2$. The $\mu^2$ dependences are cancelled between $C_i$ and $O_i$ term by term in the expansion.

The expansion in principle can serve to define the composite operators $O_i$ if one knows how to classify the short distance singularities of an operator product. In practice, however, a reverse procedure is followed in constructing the expansion: A tower of operators with right quantum numbers are first chosen, and then the coefficient functions are *calculated* by sandwiching the expansion in a set of perturbative states. Implicitly assumed in this standard procedure, though not essential to the principle of the OPE, is that the power divergences in the operators $O_i$ are subtracted with a *perturbative normal ordering*. For the operators with vacuum quantum numbers, such as $F^2 = F^{\alpha\beta}F_{\alpha\beta}$, it means the subtraction of the perturbative-vacuum expectation value. For others, it means use of the dimensional regularization in which the power divergent integrals $\int d^dk/k^\alpha = 0$ are taken to be zero.

It was first pointed out by t' Hooft [3] that the coefficient functions thus obtained, $C_i(\alpha_s) = \sum_n \alpha_s^n c_{in}$, may contain the *infrared (IR) renormalons*—a jargon for $c_{in} \sim n!$ to grow factorially with a fixed sign. Although this has never been proved conclusively in QCD except in certain limiting cases such as the number of flavors $N_f \to \infty$, many have taken the observation seriously. If true, $C_i$ is a non-Borel summable series and is genuinely infinite. If treating $C_i$ as an asymptotic series, it is easy to see the uncertainty in regularizing the series, such as truncating it around the minimal term, or regulating the singularity in the Borel plane after a Borel transformation, spoils a clear-cut classification of $C_i(\xi^2)$ in their singularities. At practical level, the power corrections to the OPE become ill-defined due to the uncertainty in the leading-order coefficient function.

For applications where power corrections are small, like heavy-quark expansion for the top flavor, the problem is of only formal importance. However, there are applications where the understanding of power corrections is absolutely essential. In extracting the strong coupling constant $\alpha_s$ from the deep-inelastic sum rules, such as Bjorken and Gross-Llewellyn

Smith sum rules, the so-called higher-twist contributions are an important source of theoretical uncertainty [4]. In the charm and bottom systems, $1/m_Q$ corrections to Isgure-Wise symmetry relations [5] valid in the heavy quark limit are substantial. Finally, the entire QCD sum rule phenomenology initiated by Shifman, Vainshtein, and Zakharov [6] relies on a meaningful understanding of power corrections at fundamental level.

The IR renormalons in QCD are generated from perturbative calculations in infrared regions of Feynman diagrams. An indication for this is that the high-order diagrams effectively renormalize the coupling constant appearing in the lower-orders. The infrared Landau singularity in the resulting running coupling is essentially the source for the $n!$ behaviors at high orders [7]. In light of this observation, the appearance of the IR renormalons in the OPE is somewhat surprising, for the OPE is devised precisely to account for soft regions of perturbative diagrams with the matrix elements of composite operators. The puzzle was solved first by F. David [8], who showed that the power-divergent composite operators defined with the standard subtraction scheme also contain the IR renormalons. And the IR renormalon singularities in the coefficient functions and operators matrix elements cancel.

Although the IR renormalons do not invalidate the standard form of the OPE, they pose practical problems in carrying out the expansion to power accuracy. The coefficient functions and the power-divergent composite operators are both ill-defined, and one must find a consistent way to regularize them. Grunberg [9] and Mueller [10] have both suggested ways to regularize the perturbation series in coefficient functions. However, computations of higher-twist contributions in their schemes appear difficult. On the other hand, in a recent work by Martinelli and Sachrajda [11], a non-perturbative renormalization of composite operators is proposed. It remains to see, however, how practical the method can be used to compute subtraction in coefficient functions.

Since the IR renormalons arise from soft regions in Feynman diagrams, a physical solution to the problem is to avoid perturbative calculations in these regions. The idea is not new, as it has been advocated by Novikov et al. [12] since many years ago. However, the only serious study in this direction was the work by Mueller [13], which has largely been ignored in the literature. The main problems with this approach, to my opinion, have been two: first, how to make systematic soft subtraction in the coefficient functions? second, how to calculate the non-perturbative matrix elements of composite operators consistent with the subtraction. In this paper, I hope to provide answers to both questions through the example of the vector-current correlation function in QCD vacuum.

To begin, let me recall once again the basic principle of the OPE: the composite operators are introduced to account for soft contributions in Feynman diagrams. Thus if the OPE is applied in a perturbative state, the composite operators shall contain perturbative soft contributions. In the standard regularization for the composite operators, however, the perturbative contributions are completely subtracted along with the power divergences. This is convenient for calculating the coefficient functions, but the operators defined in this way cannot account for the perturbative soft contributions in Feynman diagrams for correlation functions. Therefore, a fundamental solution to the IR renormalon problems is to find a practical way to regularize power divergences of composite operators while maintaining their soft contributions to a perturbative calculation.

Composite operators can be defined entirely through their insertions into Green's functions [14]. As such, the renormalization of the Green's functions determines the renormal-

ization of the composite operators. According to the standard renormalization theory, we need to consider only the Green's functions with non-negative superficial degree of divergence, which is defined for a graph $G$ as, $\delta(G) = 4 - d_O - d_\phi n_\phi$ , where $d_O$ and $d_\phi$ are the dimensions of the operator $O$ and field $\phi$, and $n_\phi$ is the number of external $\phi$ lines in the graph. The counter term operators can be constructed from the overall subtraction of the primitively divergent Green's functions. Now it is well-known that the renormalization subtraction have large freedom with finite contributions (renormalization group). In QCD, since the low momentum regions of a Feynman diagram is not perturbatively calculable, one shall not subtract divergences along with any soft contributions. In the following, I call such subtraction as *"minimal" subtraction of power divergences.*

To illustrate a systematic way of doing minimal subtraction, let me construct the composite operator $[\phi^2]$ in the $\phi^4$ theory. For our purpose, we are interested in only one insertion of the operator. According to power counting, two Green's functions have superficial divergences, $\langle 0|T\phi^2\phi(x_1)\phi(x_2)|0\rangle$ and $\langle 0|T\phi^2|0\rangle$, with superficial degrees of divergence 0 and 2, respectively. Therefore, the renormalized composite operators can be defined as,

$$[\phi^2] = Z_2\phi^2 + Z_0 I \tag{2}$$

where $Z_i$ are renormalization factors. $Z_2$ is determined by subtraction of primitive log-divergent diagrams in $\langle 0|T\phi^2\phi(x_1)\phi(x_2)|0\rangle$. On the other hand, $Z_0$ is determined by primitive quadratically-divergent diagrams in $\langle 0|T\phi^2|0\rangle$. In the standard approach, one writes, $Z_0 = -Z_2\langle 0|T\phi^2|0\rangle_{\rm pert}$. The operator $[\phi^2]$ defined in this way depends on perturbative calculations in soft regions. On the other hand, The minimal subtraction can be defined as follows. Consider any perturbative diagram for $\langle 0|TZ_2\phi^2|0\rangle$. Subtract only the contributions from regions where all the momenta in the diagrams are larger than some scale $\Lambda^2$. For instance, suppose we have the following contribution to $\langle 0|TZ_2\phi^2|0\rangle$,

$$\int_0^\infty d^4k_1 \int_0^\infty d^4k_2 ... \int_0^\infty d^4k_n f(k_1, ..., k_n) \tag{3}$$

Then we subtract the part when all the integrations run from $\Lambda^2$ to $\infty$,

$$Z_0 = -\int_{\Lambda^2}^\infty \int_{\Lambda^2}^\infty ... \int_{\Lambda^2}^\infty d^4k_1 d^4k_2 ... d^4k_n f(k_1, ..., k_n) \tag{4}$$

Thus, $Z_0$ contains only hard contributions. It remains to see however, the remaining part of the Green's functions, with mixtures of soft and hard momentum flows, are finite.

The proof for the finite remaining part is done by explicit construction. It is easy to see through working with two or higher-loop diagrams that

$$\begin{aligned}\langle 0|[\phi^2]|0\rangle_{\rm pert} = &\sum_{n=1}^\infty (-1)^{n+1} \int_0^{\mu^2} d^4k_1 ... \int_0^{\mu^2} d^4k_n \\ &\times \langle 0|TZ_2\phi^2\phi(k_1)...\phi(k_n)\phi(-k_1 - k_2 ... - k_n)|0\rangle^{\rm trun} \\ &\times \langle 0|T\phi(k_1)...\phi(k_n)\phi(-k_1 ... - k_n)|0\rangle(\mu^2)\end{aligned} \tag{5}$$

where the first factor in the integrand $\langle 0|TZ_2\phi^2\phi(k_1)...\phi(k_n)\phi(-k_1 ... - k_n)|0\rangle^{\rm trun}$ is a connected, truncated Green's function with external momenta $k_1$, $k_2$, ... ,$k_n$, $-k_1 ... - k_n$. It is finite because all logarithmic sub-divergences are cancelled by the renormalization factor $Z_2$.

The second factor $\langle 0|T\phi(k_1)...\phi(k_n)\phi(-k_1...-k_n)|0\rangle$ is an ordinary Green's function with all internal lines restricted to $k^2 > \mu^2$.

To illustrate the use of the minimal-subtracted operators in the OPE, I consider the vacuum correlation function of the conserved vector currents in the chiral limit ($m_q = 0$) $i\int e^{i\xi\cdot q}d^4\xi T[J_\mu(\xi)J_\nu(0)] = (-g_{\mu\nu}q^2 + q_\mu q_\nu)\Pi(q^2)$ . Define Adler's $D$-function through, $D(q^2) = -(2\pi)^2 q^2 (d\Pi/dq^2)$. The OPE for $D(q^2)$ at the space-like $q^2$ reads ($Q^2 = -q^2$),

$$D(Q^2) = C_0(\alpha_s, \Lambda^2))I + C_4(\alpha_s)\frac{2\pi\alpha_s F^2(\Lambda^2)}{3Q^4} + ... \ , \tag{6}$$

where $I$ is a unit operator and the renormalization scale $\mu^2$ is chosen to be $Q^2$. $F^2(\Lambda^2)$ is minimally subtracted operator without the quartic divergence,

$$F^2(\Lambda^2) = F_R^2 - \langle 0|F_R^2|0\rangle_{\rm pert}(\text{all } k^2 > \Lambda^2) \ . \tag{7}$$

The renormalized $F_R^2$ is free of logarithmic divergences. In dimensional regularization and with covariant gauge fixing, we have [15],

$$\begin{aligned} F_R^2 = {} & (1 + g\frac{\partial \ln Z_g}{\partial g})F^2 + (g\frac{\partial \ln Z_g}{\partial g} - \lambda\frac{\partial \ln Z_3}{\partial\lambda} + \frac{1}{2}g\frac{\partial \ln Z_3}{\partial g})O_{GF} \\ & + (\lambda\frac{\partial \ln \tilde{Z}}{\partial\lambda} - \frac{1}{2}g\frac{\partial \ln \tilde{Z}}{\partial g})\partial^\mu\bar{\omega}D_\mu\omega + (\lambda\frac{\partial \ln Z_2}{\partial\lambda} - \frac{1}{2}g\frac{\partial \ln Z_2}{\partial g})\bar{\psi}i\not{D}\psi \end{aligned} \tag{8}$$

where $\lambda$ is a gauge parameter, and $Z_g$, $Z_2$, $Z_3$ and $\tilde{Z}$ are renormalization constants for the gauge coupling $g$, quark fields $\psi$, gauge potential $A^\mu$ and ghost field $\omega$, respectively. All operators on the right-hand-side are unrenormalized and, $O_{GF} = -\frac{1}{2\lambda}(\partial^\mu A_\mu)^2 - \frac{1}{2}\frac{\delta S}{\delta A_\mu}A_\mu$, where $S$ is the gauge-fixed QCD action.

Let us see how the coefficient function $C_0$ and the non-perturbative matrix element $F^2(\Lambda^2)$ can be consistently calculated. First, sandwiching the expansion in Eq. (6) in the perturbative vacuum state, we have,

$$C_0(\alpha_s, \Lambda^2) = C_0(\alpha_s, 0) - C_4(\alpha_s)\frac{2\pi\alpha_s\langle 0|F^2(\Lambda^2)|0\rangle_{\rm pert}}{3Q^4} \tag{9}$$

where $C_0(\alpha_s, 0)$ is the coefficient function calculated when assuming the perturbative matrix element of $F^2(\Lambda^2)$ vanish. The contributions to $C_0(\alpha_s, 0)$ from one or two soft-gluon or ghost lines are subtracted by the second term in Eq. (9) which, according to our definition, can be computed through a formula like in Eq. (5).

Since perturbative diagrams are most conveniently evaluated in the dimensional regularization and modified minimal subtraction scheme, we shall study the OPE in this scheme (labelled by $\overline{\rm MS}$). $C_0(\alpha_s, 0)$ has been calculated to three-loops [17] (assuming three flavors),

$$C_0(\alpha_s, 0) = 1 + \frac{\alpha_s}{\pi} + 1.6\left(\frac{\alpha_s}{\pi}\right)^2 + 6.4\left(\frac{\alpha_s}{\pi}\right)^3 + ... \ , \tag{10}$$

and $C_4$ has been calculated at one-loop [18], $C_4(\alpha_s) = 1 + \frac{7\alpha_s}{6\pi} + ....$ To illustrate the soft subtraction, I have calculated the perturbative matrix element of $F^2(\Lambda^2)$ up to two-loops,

$$\langle 0|F^2(\Lambda^2)|0\rangle^{\overline{\rm MS}}_{\rm pert} = \frac{3\Lambda^4}{\pi^2}\left(1+(2.41+1.25\ln\frac{Q^2}{\Lambda^2})\frac{\alpha_s}{\pi}\right) \tag{11}$$

Finally, we have coefficient function for the unit operator in Eq. (6),

$$C_0^{\overline{\rm MS}}(\alpha_s,\Lambda^2) = 1+\frac{\alpha_s}{\pi}\left(1-2\frac{\Lambda^4}{Q^4}\right)+1.67(\frac{\alpha_s}{\pi})^2\left(1-\frac{\Lambda^4}{Q^4}\Big(2.89+1.50\ln\frac{Q^2}{\Lambda^2}\Big)\right)+... \tag{12}$$

The above series is free of the IR renormalon at $b=8\pi/\beta_0$ on the Borel plane.

To obtain the non-perturbative matrix element of $F^2(\Lambda^2)$ (gluon condensate) in dimensional regularization, we first write,

$$\langle 0|F^2(\Lambda^2)|0\rangle = \langle 0|F_R^2|0\rangle - \langle 0|F_R^2|0\rangle_{\rm pert} + \langle 0|F^2(\Lambda^2)|0\rangle_{\rm pert}\ . \tag{13}$$

According to the Joglekar-Lee theorems [16], the matrix elements of equations-of-motion operators and BRST-exact operators vanish. Thus, the first two matrix elements on the right-hand side of Eq. (13) depend on the bare operator $F^2$ only, and they transform homogeneously under a scale transformation. Furthermore, the study on the trace anomaly of the QCD energy-momentum tensor indicates $\beta(g)F^2$ is renormalization scheme and scale independent [19,20]. Thus, $\langle 0|F_R^2|0\rangle^{\overline{\rm MS}} - \langle 0|F_R^2|0\rangle^{\overline{\rm MS}}_{\rm pert} = Z\left[\langle 0|F^2|0\rangle^{\rm LAT} - \langle 0|F^2|0\rangle^{\rm LAT}_{\rm pert}\right]$ where $Z=\beta^{\rm LAT}/\beta^{\rm MS}$ depends on lattice and $\overline{\rm MS}$ $\beta$-functions. Factorizing $Z$ in Eq. (13), we find

$$\langle 0|F^2(\Lambda^2)|0\rangle^{\overline{\rm MS}} = Z\left[\langle 0|F^2|0\rangle^{\rm LAT} - \langle 0|F^2|0\rangle^{\rm LAT}_{\rm pert} + Z^{-1}\langle 0|F^2(\Lambda^2)|0\rangle^{\overline{\rm MS}}_{\rm pert}\right] \tag{14}$$

The above is a practical definition of the gluon condensate that are renormalon-free and consistent with the $\overline{\rm MS}$ evaluation of the coefficient functions.

The first two terms in Eq. (14) are what has been used as a lattice definition of the gluon condensate in the literature [21]. However, $\langle 0|F^2|0\rangle^{\rm LAT}_{\rm pert}$ is an ill-defined perturbation series due to the IR renormalon at $b=8\pi/\beta_0$. The perturbation series has been numerically evaluated up to eight-loops by Di Renzo et al. in the quenched approximation [22] (see also [23],

$$\langle 0|F^2|0\rangle^{\rm LAT}_{\rm pert} = \frac{48}{a^4}\left(1+4.01\frac{\alpha_s(a)}{\pi}+64.08\Big(\frac{\alpha_s(a)}{\pi}\Big)^2+1321.86\Big(\frac{\alpha_s(a)}{\pi}\Big)^3+...\right) \tag{15}$$

To extract the gluon condensate with a good accuracy, one is faced with two opposite requirements for the lattice spacing $a$. To ensure that the calculations are at the continuum limit, one shall take as small $a$ as possible. On the other hand, for small $a$, the cancellation of quartic divergences requires extremely good accuracy in the perturbation series. Even for $a=0.17$ fm ($\beta=5.7$), one needs to evaluate the perturbation series to at least one percent. High-accuracy is not possible without regularizing the IR renormalon in the series. [Although for expansions with the bare lattice coupling, the smallest term is postponed to higher orders, the intial decrease of the terms is very slow due to large tadpole contributions.] The last term in Eq. (14) is introduced precisely for the IR renormalon regularization. Using the one-loop relation between the lattice and $\overline{\rm MS}$ couplings [24], I find,

$$Z^{-1}\langle 0|F^2(\Lambda^2)|0\rangle^{\overline{\rm MS}}_{\rm pert} = \frac{3\Lambda^4}{\pi^2}\left(1+\Big(20.87+1.25\ln\frac{Q^2}{\Lambda^2}+2.75\ln(Q^2a^2)\Big)\frac{\alpha_s(a)}{\pi}+...\right) \tag{16}$$

To determined the subtraction up to three-loops, we need two-loop relations between lattice and $\overline{\rm MS}$ couplings and the matrix element $\langle 0|F^2(\Lambda^2)|0\rangle^{\overline{\rm MS}}_{\rm pert}$ up to three-loops, the former has recently been calculated by Luscher and P. Weisz [25]. We wish to make a three-loop subtraction calculation in the near future.

To summarize, I have introduced a minimal subtraction scheme to regularize power divergences of composite operators. When these operators are used in Wilson's OPE, the coefficient functions are automatically free of the IR renormalon singularities. The matrix elements of the composite operators can be calculated in non-perturbative methods such as lattice QCD. Thus, within the scheme one can meaningfully discuss power corrections to desired accuracy. The scheme can be straightforwardly applied to deep-inelastic sum rules where the mixing of higher-twist operators with lower-twist ones is an important issue [26]. It can also be used in the heavy quark effective theory to define the pole masse for the heavy-quark expansion [27]. This will be discussed in a separate publication.


## ACKNOWLEDGMENTS

I wish to thank N. Christ, A. Mueller, J. Negele, U. Wiese for useful conversations about the subject.

## REFERENCES


[1] K. G. Wilson, Phys. Rev. 179 (1969) 1499.

[2] See for example, T. Muta, *Fundations of Quantum Chromodynamcs*, World Scientific, Singapore, 1987.

[3] G. 't Hooft, in: The whys of subatomic physics (Erice, 1977), ed. A. Zichichi (Plenum, New York, 1977).

[4] X. Ji, talk presented at the Yale symposium on the spin structure of the nucleon, Jan., 1994.

[5] N. Isgur and M. Wise, Phys. Lett. B232 (1989) 113; B237 (1990) 527.

[6] M. A. Shifman, A. I. Vainshtein, and V. I. Zhakharov, Nucl. Phys. B147 (1979) 385, 448.

[7] V. I. Zakharov, Nucl. Phys. B385 (1992) 452.

[8] F. David, Nucl. Phys. B234 (1984) 237; B263 (1986) 637.

[9] G. Grunberg, Phys. Letts. B325 (1994) 441.

[10] A. H. Mueller, Phys. Letts. B308 (1993) 355.

[11] G. Marinelli and C. T. Sachrajda, CERN-TH.7517/94, SHEP 94/95-13, Feb. 1995.

[12] V. A. Novikov et. al. Nucl. Phys. B249 (1985) 445.

[13] A. H. Mueller, Nucl. Phys. B250 (1985) 327.

[14] W. Zimmermann, Ann. Phys. (N.Y.), 1971.

[15] X. Ji, MIT-CTP Preprint No. 2407, 1995.

[16] S. D. Joglekar and B. W. Lee, Ann. Phys. (N. Y.) 97 (1976) 160.

[17] S. G. Gorishny, A. L. Kataev, and S. A. Larin, Phys. Lett. B259 (1991) 144.

[18] K. G. Chetyrkin, S. G. Gorishny, and V. P. Spiridonov, Phys. Lett. B160 (1985) 149.

[19] J. C. Collins, A. Duncan, and S. D. Joglekar, Phys. Rev. D16 (1977) 438.

[20] S. Caracciolo, P. Menotti, and A. Pelissetto, Nucl. Phys. B375 (199) 195.

[21] T. Banks et al., Nucl. Phys. B190 (1981) 692; A. DiGiacomo and Rossi, Phys. Lett. 100B (1981) 481, 108B(1982) 327.

[22] F. Di Renzo, E. Onofri, and G. Marchesini, UPRF-95-418, IFUM 540-FT, hep-th/9505095.

[23] B. Alles, M. Campostrini, A. Feo, and H. Panagopoulos, Phys. Lett. B324 (1994) 433.

[24] A. Hasenfratz and P. Hasenfratz, Phys. Lett. B93 (1980) 165.

[25] M. Luscher and P. Weisz, MPI-PHT-95-40, hep-lat/9595011.

[26] X. Ji, MIT-CTP Preprint No. 2381, 1994.

[27] See many refrences in V. Braun and M. Beneke, Nucl. Phys. B426 (1994) 301.